

# ON THE BEST APPROXIMATION OF THE INFINITESIMAL GENERATOR OF A CONTRACTION SEMIGROUP IN A HILBERT SPACE<sup>1</sup>

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**Abstract:** Let  $A$  be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space  $H$ . We give an upper estimate for the best approximation of the operator  $A$  by bounded linear operators with a prescribed norm in the space  $H$  on the class  $Q_2 = \{x \in \mathcal{D}(A^2) : \|A^2x\| \leq 1\}$ , where  $\mathcal{D}(A^2)$  denotes the domain of  $A^2$ .

**Key words:** Contraction semigroup, Infinitesimal generator, Stechkin's problem.

## 1. Introduction

Let  $H$  be a Hilbert space with the inner product  $(\cdot, \cdot)$  and the norm  $\|\cdot\|$ , and let  $A$  be the infinitesimal generator of a strongly continuous contraction semigroup in  $H$ . For the definition and properties of the infinitesimal generator of a semigroup in a Banach space see, e.g., [6, §14.2]. Note that a strongly continuous contraction semigroup is also called a contraction semigroup of the class  $C_0$  ([8, 9]). For an operator  $F$  on the space  $H$ ,  $\mathcal{D}(F)$  denotes the domain of  $F$ . We denote by  $I$  the identity operator.

In this paper, we study the so-called Stechkin's problem of the best approximation of the operator  $A$  by bounded linear operators with a prescribed norm on the class of elements  $x \in \mathcal{D}(A^2)$  such that  $\|A^2x\| \leq 1$ . We give an upper estimate for the best approximation of the operator  $A$ .

The problem we consider is a special case of the general problem of the best approximation of an unbounded operator by linear bounded ones on a certain class of elements in a Banach space. This problem first appeared in Stechkin's work in 1965–1967 [11]. The problem was studied by a number of authors (see surveys [1], [2], monograph [4], paper [3], and the bibliography therein).

Stechkin formulated this problem in a general setting as follows. Let  $X, Y$  be two Banach spaces, let  $A$  be a linear operator (in general, unbounded) from  $X$  to  $Y$ , and let  $Q \subseteq \mathcal{D}(A)$  be a certain class of elements from the domain  $\mathcal{D}(A)$  of the operator  $A$ . We denote by  $\mathcal{B}(N)$  the set of

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linear bounded operators from  $X$  to  $Y$  with the norm  $\|T\|_{X \rightarrow Y} \leq N$ . The best approximation of the operator  $A$  by linear bounded operators  $T \in \mathcal{B}(N)$  on the class  $Q$  is

$$E_N(A; Q) = \inf \{U(A, T, Q) : T \in \mathcal{B}(N)\},$$

where

$$U(A, T, Q) = \sup \{\|Ax - Tx\|_Y : x \in Q\}$$

is the deviation of the operator  $T$  from the operator  $A$  on the class  $Q$ .

One of the most important cases of the problem formulated above is when the class  $Q$  is defined in the following way. Let  $Z$  be a Banach space and  $B$  be a linear operator from  $X$  to  $Z$  such that  $\mathcal{D}(B) \subseteq \mathcal{D}(A)$ . The class  $Q$  is then defined as  $Q = \{x \in X : \|Bx\|_Z \leq 1\}$ .

Stechkin [11] suggested an estimate from below for the best approximation  $E_N(A; Q)$  in terms of the modulus of continuity of the operator  $A$  on the class  $Q$  defined by

$$\Phi(\delta) = \sup \{\|Ax\|_Y : x \in Q, \|x\|_X \leq \delta\}, \quad \delta > 0.$$

Namely, Stechkin showed that

$$E_N(A; Q) \geq \sup \{\Phi(\delta) - N\delta : \delta > 0\}. \quad (1.1)$$

In particular, when  $B = A^n$ , the problem  $E_N(A^k; Q)$  turned out to be closely connected to the exact constants in the Kolmogorov-type inequalities of the form

$$\|A^k x\| \leq C \|x\|^{\frac{n-k}{n}} \|A^n x\|^{\frac{k}{n}}, \quad x \in \mathcal{D}(A^n), \quad (1.2)$$

with  $n, k \in \mathbb{N}$ ,  $0 < k < n$ , and a certain constant  $C$  that depends on  $n$  and  $k$ .

If  $A$  is the differentiation operator, inequalities (1.2) are inequalities between the norms of the derivatives of a function. Such inequalities have been studied by a large number of authors (see [1], [2], [4] and the bibliography therein). Here we only mention that Hardy, Littlewood and Pólya [7, Chapter VII, §7.8] obtained the exact inequality

$$\|f'\|^2 \leq 2\|f\|\|f''\| \quad (1.3)$$

in the space  $L_2(0, \infty)$  on the class of functions  $f \in L_2(0, \infty)$  such that  $f'$  is locally absolutely continuous on  $(0, \infty)$ , and  $f'' \in L_2(0, \infty)$ .

In 1971, Kato [9] proved the following result which can be considered as a generalization of (1.3). Let  $A$  be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space  $H$ . Then

$$\|Ax\|^2 \leq 2\|x\|\|A^2x\|, \quad x \in \mathcal{D}(A^2).$$

In this paper, we study Stechkin's problem of the best approximation of the infinitesimal generator  $A$  of a strongly continuous contraction semigroup by bounded linear operators on the class

$$Q_2 = \{x \in \mathcal{D}(A^2) : \|A^2x\| \leq 1\} \quad (1.4)$$

in a Hilbert space. Namely, we estimate

$$E_N(A; Q_2) = \inf \{U(T) : T \in \mathcal{B}(N)\}, \quad (1.5)$$

where

$$U(T) = U(A, T, Q_2) = \sup \{\|Ax - Tx\| : x \in Q_2\}. \quad (1.6)$$

## 2. The main result

The main result of the paper is the following statement.

**Theorem 1.** *The best approximation (1.5) of the infinitesimal generator  $A$  of a strongly continuous contraction semigroup in a Hilbert space on the class  $Q_2$  defined in (1.4) satisfies the inequality*

$$E_N(A; Q_2) \leq \frac{1}{N}.$$

It is known that the infinitesimal generator  $A$  of a strongly continuous contraction semigroup in a Banach space possesses the following properties:

- 1) The domain  $\mathcal{D}(A)$  of the operator  $A$  is dense (see, e.g., [6, Lemma 14.5, p. 411]).
- 2) The resolvent set  $\rho(A)$  of the operator  $A$  contains the right half-plane  $\{\lambda \in \mathbb{C} \mid \Re \lambda > 0\}$ . Moreover,  $\|(A - \lambda I)^{-1}\| \leq (\Re \lambda)^{-1}$  for all  $\lambda \in \mathbb{C}$  with  $\Re \lambda > 0$  (e.g., [6, Theorem 14.7, p. 412]).

Furthermore, if  $A$  is the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space, we have additionally:

- 3) The operator  $A$  is upper semibounded, with the upper bound 0, i.e.,

$$\Re(Ax, x) \leq 0$$

for  $x \in \mathcal{D}(A)$  [6, Lemma 14.9, p. 416].

The following lemma is not new. However, we will formulate and prove it for the sake of completeness.

**Lemma 1.** *Let  $A$  be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space  $H$  and  $c > 0$ . Then the operator*

$$B_c = (cI + A)(cI - A)^{-1}$$

*is densely defined and bounded (and thus can be extended to the whole space  $H$  by continuity). Moreover,*

$$\|B_c\| \leq 1.$$

*Remark.* The operator  $B_c$  is the Cayley transform of the operator  $A$  in the terminology of Kato [9], see also [10, p. 545].

**P r o o f.** Since  $c > 0$ , the operator  $(cI - A)^{-1}$  is defined everywhere on  $H$  and bounded. Since  $A$  is the infinitesimal generator of a strongly continuous contraction semigroup, the operator  $-A$  is  $m$ -accretive (see [10, Chapter IX, §1.4 as well as Problem 1.18, both p. 485]). Therefore, the domain  $\mathcal{D}(A)$  of the operator  $A$  is equal to the range  $\mathcal{R}((cI - A)^{-1})$  of the operator  $(cI - A)^{-1}$  which is dense in  $H$  (see [10, Chapter V, §3.10, p. 279]). Thus,  $B_c$  is densely defined.

Now we estimate the norm of  $B_c$ . For  $x \in \mathcal{D}(A)$  we have

$$\|cx + Ax\|^2 = c^2\|x\|^2 + \|Ax\|^2 + 2c\Re(Ax, x),$$

$$\|cx - Ax\|^2 = c^2\|x\|^2 + \|Ax\|^2 - 2c\Re(Ax, x).$$

It follows immediately that

$$\|(cI + A)x\| \leq \|(cI - A)x\|. \quad (2.1)$$

Now take  $y \in \mathcal{D}((cI - A)^{-1})$ . Applying (2.1) to  $x = (cI - A)^{-1}y \in \mathcal{D}(A)$ , we obtain

$$\|(cI + A)(cI - A)^{-1}y\| \leq \|y\|,$$

and thus  $\|B_c\| \leq 1$ . □

Now we are ready to prove Theorem 1.

**P r o o f.** We will construct a concrete approximating operator  $T$  in problem (1.5) and estimate its norm and its deviation (1.6) from the operator  $A$  on the class  $Q_2$ .

Note that all the operators we consider commute on the set  $\mathcal{D}(A^2)$ .

The restriction of the operator  $A$  to the set  $\mathcal{D}(A^2)$  (which we will denote by the same symbol) can be represented as

$$A = \frac{N}{2}(B_N - I) - \frac{1}{2N}(B_N + I)A^2.$$

Put  $T : H \rightarrow H$ ,

$$T = \frac{N}{2}(B_N - I).$$

Then, for the restriction of the operator  $A - T$  to  $\mathcal{D}(A^2)$ , we have

$$A - T = -\frac{1}{2N}(B_N + I)A^2.$$

We estimate the norm of the operator  $T$  as follows:

$$\|T\| = \frac{N}{2}\|B_N - I\| \leq \frac{N}{2}(\|B_N\| + \|I\|) = N. \quad (2.2)$$

For the deviation  $U(T)$  of the operator  $T$  from the operator  $A$ , we obtain that

$$U(T) = \sup_{x \in Q_2} \|(A - T)x\| \leq \sup_{x \in Q_2} \frac{1}{2N}\|B_N + I\| \cdot \|A^2x\| \leq \frac{1}{N}. \quad (2.3)$$

It follows immediately from (2.2) and (2.3) that

$$E_N(A; Q_2) \leq U(T) \leq \frac{1}{N}.$$

□

### 3. Approximation of the differentiation operator in the space $L_2(0, \infty)$

An important concrete case of problem (1.5) is the problem of the best approximation of the differentiation operator  $Df = f'$  by bounded linear operators in the Hilbert space  $L_2(0, \infty)$  of real-valued functions whose squares are integrable on  $(0, \infty)$  on the class  $Q^{(2)}$  defined as follows:

$Q^{(2)}$  is the class of functions  $f \in L_2(0, \infty)$  such that  $f'$  is locally absolutely continuous on  $[0, \infty)$ ,  $f'' \in L_2(0, \infty)$ , and  $\|f''\| \leq 1$ . Problem (1.5) takes in this case the form

$$E_N(D; Q^{(2)}) = \inf_{T \in \mathcal{B}(N)} \sup_{f \in Q^{(2)}} \|f' - Tf\|. \quad (3.1)$$

It took about 20 years of research to solve the problem completely. Stechkin's inequality (1.1) and inequality (1.3) of Hardy, Littlewood and Pólya provide the lower bound

$$E_N(D; Q^{(2)}) \geq \frac{1}{2N}.$$

One of the first upper bounds for (3.1)

$$E_N(D; Q^{(2)}) \leq \frac{1}{\sqrt{3}N}$$

was obtained by using a concrete approximating operator by the first named author in 1996 [5]. Problem (3.1) was fully solved only in 2014 by Arestov and the second named author [3]. Namely, they showed that

$$E_N(D; Q^{(2)}) = \frac{1}{2N}.$$

In this section, we discuss what the statement of Theorem 1 means in the concrete case (3.1) of problem (1.5). The approximating operator  $T$  used in Theorem 1 is

$$T = \frac{N}{2}(B_N - I) = NA(NI - A)^{-1}. \quad (3.2)$$

Below we will describe this operator in the special case. We consider and calculate its norm  $\|T\|$  and its deviation  $U(T)$  from the operator  $A = D$  on the class  $Q^{(2)}$ .

It is not difficult to see that the operator  $T$  in the concrete case can be represented as follows. Let  $W$  be the class of functions  $y \in L_2(0, \infty)$  such that  $y$  is locally absolutely continuous on  $[0, \infty)$  and  $y' \in L_2(0, \infty)$ . For  $f \in L_2(0, \infty)$ , we consider the differential equation

$$-y' + Ny = f, \quad y \in W. \quad (3.3)$$

For each function  $f \in L_2(0, \infty)$ , equation (3.3) has a unique solution which is a real-valued function from  $L_2(0, \infty)$ . The operator  $T$  is defined as

$$Tf = Ny', \quad (3.4)$$

where  $y$  is the solution of the differential equation (3.3).

Integrating by parts and taking into account that  $\lim_{t \rightarrow \infty} y(t) = 0$ , we obtain (see [3] for details) that

$$\|f\|^2 = \int_0^\infty (-y'(t) + Ny(t))^2 dt = \int_0^\infty (y'(t))^2 dt + N^2 \int_0^\infty (y(t))^2 dt + Ny^2(0).$$

It follows from (3.4) that  $\|Tf\|^2 = N^2 \int_0^\infty (y'(t))^2 dt$ . Thus, we immediately obtain

$$\|Tf\|^2 \leq N^2 \|f\|^2, \quad (3.5)$$

which gives the estimate  $\|T\| \leq N$ . Now we show that indeed  $\|T\| = N$ . Consider the family of functions  $y_K = e^{-Kt}$ ,  $K > 0$ . Let  $f_K$  be the corresponding right-hand side of equation (3.3). Take an arbitrary  $0 < \alpha < 1$ . We have

$$\begin{aligned} \alpha N^2 \|f_K\|^2 - \|Tf_K\|^2 &= \alpha N^2 \int_0^\infty (-y'_K(t) + Ny_K(t))^2 dt - N^2 \int_0^\infty (y'_K(t))^2 dt \\ &= \frac{N^2}{2K} (\alpha(K+N)^2 - K^2). \end{aligned}$$

This expression is negative for all  $0 < \alpha < \frac{K^2}{(N+K)^2}$  which yields  $\|Tf_K\|^2 > \alpha N^2 \|f_K\|^2$ . Letting  $K$  go to infinity (with fixed  $N$ ) we let  $\alpha$  approach 1, and thus obtain  $\|T\| \geq N$ . Consequently,  $\|T\| = N$ .

Note that inequality (3.5) is a strict inequality if  $y \neq 0$  and, consequently,  $f \neq 0$ . In other words, the norm of the operator  $T$  is not attained.

It can be shown similarly that the norm of the operator  $V = -\frac{1}{2N}(B_N + I)$  is equal to  $1/N$ . Since the domain  $\mathcal{D}(D^2)$  of the operator  $D^2$  is dense in  $L_2(0, \infty)$ , it follows that the deviation of the operator  $T$  from the differentiation operator  $D$  on the class  $Q^{(2)}$  is equal to  $1/N$ .

Thus, the approximating operator (3.2) gives the estimate  $E_N(D; Q^{(2)}) \leq \frac{1}{N}$  in the general case (1.5) as well as in the concrete case (3.1).

## REFERENCES

1. **Arestov V.V.** Approximation of unbounded operators by bounded operators and related extremal problems // Russian Math. Surveys, 1996. Vol. 51, no. 6. P. 1093–1126. DOI: 10.4213/rm1019
2. **Arestov V.V., Gabushin V.N.** Best approximation of unbounded operators by bounded operators // Russian Math. (Iz. VUZ), 1995. Vol. 39, no. 11. P. 38–63.
3. **Arestov V.V., Filatova M.A.** Best approximation of the differentiation operator in the space  $L_2$  on the semiaxis // J. Approx. Theory, 2014. Vol. 187. P. 65–81. DOI: 10.1016/j.jat.2014.08.001
4. **Babenko V.F., Korneichuk N.P., Kofanov V.A., Pichugov S.A.** Inequalities for derivatives and their applications. Kiev: Naukova Dumka, 2003. [in Russian]
5. **Berdysheva E.E.** On the best approximation of the differentiation operator in  $L_2(0, \infty)$  // East J. Approx. 1996. Vol. 2, no. 3. P. 281–287.
6. **Grubb G.** Distributions and Operators. NY: Springer Science and Business Media, 2008. 464 p. DOI: 10.1007/978-0-387-84895-2
7. **Hardy G.H., Littlewood J.E., Pólya G.** Inequalities. Cambridge University Press, 1934. 314 p.
8. **Hille E., Phillips R.S.** Functional Analysis and Semigroups. American Mathematical Society, 1957. 808 p.
9. **Kato T.** On an inequality of Hardy, Littlewood and Pólya // Adv. Math., 1971. Vol. 7. P. 217–218. DOI: 10.1016/S0001-8708(71)80002-6
10. **Kato T.** Perturbation Theory for Linear Operators. Verlag–Berlin–Heidelberg: Springer, 1995. 623 p. DOI: 10.1007/978-3-642-66282-9
11. **Stechkin S.B.** Best approximation of linear operators // Math. Notes. 1967. Vol. 1. P. 91–99. DOI: 10.1007/BF01268056